THE DIFFEOTOPY GROUP OF $S^1 \times S^2$ VIA CONTACT TOPOLOGY

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Abstract. As shown by H. Gluck in 1962, the diffeotopy group of $S^1 \times S^2$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Here an alternative proof of this result is given, relying on contact topology. We then discuss two applications to contact topology: (i) it is shown that the fundamental group of the space of contact structures on $S^1 \times S^2$, based at the standard tight contact structure, is isomorphic to $\mathbb{Z}$; (ii) inspired by previous work of M. Fraser, an example is given of an integer family of Legendrian knots in $S^1 \times S^2 \# S^1 \times S^2$ (with its standard tight contact structure) that can be distinguished with the help of contact surgery, but not by the classical invariants (topological knot type, Thurston–Bennequin invariant, and rotation number).

1. Introduction

The *diffeotopy group* $\mathcal{D}(M)$ of a smooth manifold M is the quotient of the diffeomorphism group $\mathrm{Diff}(M)$ by its normal subgroup $\mathrm{Diff}_0(M)$ of diffeomorphisms isotopic to the identity. Alternatively, one may think of the diffeotopy group as the group $\pi_0(\mathrm{Diff}(M))$ of path components of $\mathrm{Diff}(M)$, since any continuous path in $\mathrm{Diff}(M)$ can be approximated by a smooth one, i.e. an isotopy. We use this terminology to emphasise that we work in the differentiable category throughout. In the topological realm, with diffeomorphisms replaced by homeomorphisms, one speaks of the *homeotopy group*. In either situation, the more popular term is *mapping class group*, sometimes with the attribute 'extended' in order to indicate that orientation-reversing maps are allowed.

Quite a bit is known about the diffeotopy groups of 3-manifolds. The theorem of Cerf [4] says that $\mathcal{D}(S^3) = \mathbb{Z}_2$. The diffeotopy groups of lens spaces were computed independently by Bonahon [1] and Hodgson–Rubinstein [26]. For other known results, open questions, and an extensive bibliography, see Kirby's problem list [29], especially Problems 3.34 to 3.36.

The diffeotopy group of $S^1 \times S^2$ was determined by Gluck [22]. He showed that $\mathcal{D}(S^1 \times S^2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. (Actually, Gluck dealt with the homeotopy group, but in dimension 3 this amounts to the same; see the discussion in Section 5.8 of [26].) Our aim in this note is to derive that result by contact topological means. The main ingredients are the classification of contact structures on $S^1 \times S^2$ up to isotopy, a result of Colin about isotopies of 2-spheres in contact 3-manifolds, and a theorem of Giroux concerning the space of contact elements on $\mathbb{R}^2$ and its contactomorphism group. This may indicate to what extent one might hope to generalise our method to other 3-manifolds.

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The key point in the determination of the diffeotopy group of $S^1 \times S^2$ is to show that any diffeomorphism acting trivially on homology is isotopic to either the identity or a diffeomorphism r of order 2 (up to isotopy) that will be described in the next section. This argument will take up Sections 3 to 7.

We then put this result to use in contact topology. In Section 8 we show that the fundamental group of the space of contact structures on $S^1 \times S^2$, based at the standard tight contact structure ξ_{st}, is isomorphic to $\mathbb{Z}$. This follows essentially from the observation that the mentioned diffeomorphism r is isotopic to a contactomorphism r_{c} of infinite order in the contactomorphism group (as was noticed previously by Gompf [23]).

In Section 9 we give an explicit description of an infinite family of homologically trivial Legendrian knots in $(S^1 \times S^2 \# S^1 \times S^2, \xi_{\mathrm{st}} \# \xi_{\mathrm{st}})$, all of which have the same topological knot type, Thurston–Bennequin invariant, and rotation number, but which are pairwise not Legendrian isotopic. This family has previously been described by Fraser, albeit in an implicit fashion only. Moreover, we shall explain why we regard her argument as incomplete.

2. The diffeotopy group of $S^1 \times S^2$

Given a manifold M, write $\mathrm{Aut}_i(M)$ for the group of automorphisms of the homology group $H_i(M)$. We consider the homomorphism

$$\begin{array}{rcccccc} \Phi\colon & \mathcal{D}(S^1 \times S^2) & \longrightarrow & \mathrm{Aut}_1(S^1 \times S^2) & \oplus & \mathrm{Aut}_2(S^1 \times S^2) \\ & [f] & \longmapsto & (f_*|_{H_1} & , & f_*|_{H_2}). \end{array}$$

Since $H_i(S^1 \times S^2) \cong \mathbb{Z}$ for $i = 1, 2$, this gives a homomorphism $\Phi\colon \mathcal{D}(S^1 \times S^2) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$. For the interpretation of $\mathbb{Z}_2$ as the automorphism group of $\mathbb{Z}$ it is convenient to write $\mathbb{Z}_2$ multiplicatively with elements ± 1. In order to study the properties of Φ, we introduce the following diffeomorphisms.

Write r_θ for the rotation of $S^2 \subset \mathbb{R}^3$ about the x_3-axis through an angle θ. We think of S^1 as $\mathbb{R}/2\pi\mathbb{Z}$. Define diffeomorphisms s, a, r of $S^1 \times S^2$ by

$$\begin{array}{rcl} s(\theta, \mathbf{x}) & = & (-\theta, \mathbf{x}), \\ a(\theta, \mathbf{x}) & = & (\theta, -\mathbf{x}), \\ r(\theta, \mathbf{x}) & = & (\theta, r_\theta(\mathbf{x})). \end{array}$$

Then $\Phi([s]) = (-1, 1)$, $\Phi([a]) = (1, -1)$, and $\Phi([r]) = (1, 1)$. So Φ is surjective, and — since s and a commute with each other — a splitting of Φ can be defined by sending $(1, 1)$ to $[\mathrm{id}_{S^1 \times S^2}]$, the element $(-1, 1)$ to $[s]$, and $(1, -1)$ to $[a]$. We therefore have a split short exact sequence

$$\ker \Phi \rightarrowtail \mathcal{D}(S^1 \times S^2) \twoheadrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Lemma 1. *The class $[r]$ has order 2 in $\mathcal{D}(S^1 \times S^2)$.*

Proof. The fact that the order of $[r]$ is at most 2 follows from $r^2(\theta, \mathbf{x}) = (\theta, r_{2\theta}(\mathbf{x}))$ and $\pi_1(\mathrm{SO}_3) = \mathbb{Z}_2$. Actually, this shows that r^2 is isotopic to the identity via an isotopy preserving the S^2-leaves in the product foliation of $S^1 \times S^2$.

In order to show that r is not isotopic to the identity, we choose a trivialisation of the tangent bundle $T(S^1 \times S^2)$ by an oriented frame. We may assume that along $S^1 \equiv S^1 \times \{(0, 0, 1)\}$ this frame is $\partial_\theta, \partial_{x_1}, \partial_{x_2}$. Then any orientation-preserving diffeomorphism f of $S^1 \times S^2$ induces an element $[Tf|_{S^1}] \in \pi_1(\mathrm{GL}_3^+)$. Isotopic

diffeomorphisms induce the same element. The identity on $S^1 \times S^2$ induces the trivial element; the diffeomorphism r, the non-trivial element in $\pi_1(\mathrm{GL}_3^+) = \mathbb{Z}_2$. $\square$

Our main goal will be to prove the following statement.

Proposition 2. *The subgroup* $\ker \Phi \subset \mathcal{D}(S^1 \times S^2)$ *is generated by* $[r]$*, and hence isomorphic to* $\mathbb{Z}_2$*. In other words, any diffeomorphism of* $S^1 \times S^2$ *acting trivially on homology is isotopic to either* id *or* r*.*

The result $\mathcal{D}(S^1 \times S^2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is an immediate consequence: from the split short exact sequence above we know that $\mathcal{D}(S^1 \times S^2)$ is the semidirect product of the normal subgroup $\mathbb{Z}_2$ and the quotient $\mathbb{Z}_2 \oplus \mathbb{Z}_2$; but a normal subgroup of order 2 is central, so the action of the quotient by conjugation is trivial.

Thus, let f be a diffeomorphism of $S^1 \times S^2$ acting trivially on homology. (In particular, f preserves the orientation.) The strategy will be to isotope f step by step to a diffeomorphism satisfying a number of additional properties, until we arrive at id or r. After each step, we continue to write f for the new diffeomorphism.

3. From a diffeomorphism to a contactomorphism

We shall rely freely on some fundamental notions and results from contact topology, all of which can be found in [18].

The standard tight contact structure ξ_{st} on $S^1 \times S^2 \subset S^1 \times \mathbb{R}^3$ is given by

$$\alpha := x_3\, d\theta + x_1\, dx_2 - x_2\, dx_1 = 0.$$

This is the unique positive tight contact structure on $S^1 \times S^2$ up to isotopy, see [18, Thm. 4.10.1].

Lemma 3. *The diffeomorphism* f *is isotopic to a contactomorphism*[1] *of* ξ_{st}*.*

Proof. The contact structure $Tf(\xi_{\mathrm{st}})$, which is again positive and tight, is isotopic to ξ_{st}. Gray stability [18, Thm. 2.2.2] then gives the desired isotopy. $\square$

Later on we shall need a contactomorphism r_{c} representing the class $[r]$ (there should be no confusion with the notation r_θ used earlier). There are two ways of exhibiting such a contactomorphism: the first one uses the above description of $(S^1 \times S^2, \xi_{\mathrm{st}})$; the second one is better adapted to describing the effect on Legendrian curves in the front projection picture.

A straightforward computation yields

$$r^*\alpha = (x_3 + x_1^2 + x_2^2)\, d\theta + x_1\, dx_2 - x_2\, dx_1.$$

We claim that r can be isotoped to a contactomorphism r_{c} by an isotopy that shifts each 2-sphere $\{\theta\} \times S^2$ along its characteristic foliation induced by ξ_{st}. Indeed, that foliation is given by the vector field

$$X = x_1 x_3 \partial_{x_1} + x_2 x_3 \partial_{x_2} + (x_3^2 - 1)\partial_{x_3},$$

with singular points at $(x_1, x_2, x_3) = (0, 0, \pm 1)$. One computes

$$L_X \alpha = i_X d\alpha = (x_3^2 - 1)\, d\theta + 2x_1 x_3\, dx_2 - 2x_2 x_3\, dx_1 = 2x_3\alpha - (1 + x_3^2)\, d\theta.$$

This shows that the flow of X has the desired effect of decreasing the $d\theta$-component relative to the dx_1- and dx_2-components; thus, a suitable rescaling of X by a

[1] Contact structures are assumed to be cooriented; contactomorphisms are understood to preserve the coorientation.

function that depends only on x_3 will give us a flow that moves the contact structure $\ker(r^*\alpha)$ back to $\ker\alpha = \xi_{\mathrm{st}}$.

An alternative picture, due to Gompf [23, p. 636], is based on a contactomorphism

$$\bigl(S^1 \times (S^2 \setminus \{\text{poles}\}), \xi_{\mathrm{st}}\bigr) \cong \bigl(\mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})^2, \ker(dz + x\, dy)\bigr).$$

The 2-spheres $\{\theta_0\} \times S^2$ correspond to the annuli $\{y = y_0\}$, each compactified by two points at $x = \pm\infty$. Now a simple description of a contactomorphism r_{c} in the class $[r]$ is given by a Dehn twist along a circle $\{y = y_0\}$ in the torus $(\mathbb{R}/2\pi\mathbb{Z})^2$ (plus a shift in the x-direction to make it a contactomorphism):

$$(x, y, z) \longmapsto (x - 1, y, y + z).$$

Figure 1 shows the effect of that Dehn twist on the Legendrian circle $t \mapsto (0, t, 0)$ in $\mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})^2$, followed by a Legendrian isotopy corresponding to Gompf's 'move 6', of which we shall see more in the next section. (The figure shows the front projection to the yz-torus.)

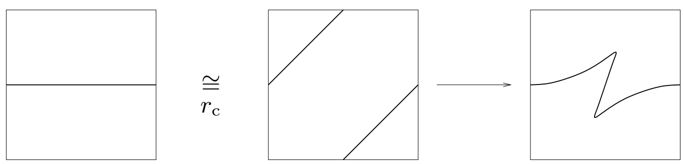

FIGURE 1. The contactomorphism r_{c}, followed by 'move 6'.

4. FIXING A 2-SPHERE

As described in [23, Section 2] or [24, Section 11.1], cf. [12], one can represent the contact manifold $(S^1 \times S^2, \xi_{\mathrm{st}})$ by the Kirby diagram with one 1-handle only (Figure 2) in the standard contact structure on S^3. The attaching balls for the 1-handle are drawn as round balls, but it is understood that these balls are in fact chosen in such a way that the characteristic foliation on their boundary induced by the standard contact structure on S^3 is the same as the characteristic foliation on $\{\theta\} \times S^2$ induced by ξ_{st}.

FIGURE 2. $S^1 \times S^2$ with its standard tight contact structure.

Lemma 4. *The contactomorphism f is contact isotopic to a contactomorphism fixing a sphere $S_0 := \{0\} \times S^2$, which we think of as the boundary of the attaching balls in Figure 2.*

Proof. Since f is a contactomorphism, ξ_{st} induces the same characteristic foliation on $f(S_0)$ as on S_0. As shown by Colin [9], with ξ_{st} being tight this implies that $f(S_0)$ and S_0 are contact isotopic — and hence f contact isotopic to a contactomorphism fixing S_0 —, provided the two 2-spheres are topologically isotopic.

For showing the existence of such a topological isotopy, we essentially rely on stage one of Gluck's proof [22]; the argument is included here for the reader's convenience.

If $S_1 := f(S_0)$ is disjoint from S_0, then those two spheres bound a compact manifold that constitutes an h-cobordism W between them. This follows from the fact that the two spheres are homotopic; recall that f acts trivially on $H_2(S^1 \times S^2) = \pi_2(S^1 \times S^2)$. This h-cobordism W is contained inside $S^1 \times S^2$ and therefore does not contain any fake 3-cells — without appeal to Perelman's positive answer to the Poincaré conjecture. It follows that W is diffeomorphic to $S^2 \times [0,1]$, and hence S_0 isotopic to S_1. (This argument is due to Laudenbach [30].)

In the general case, we first use an isotopy to bring S_0 and S_1 into general position, such that they intersect transversely in a finite number of circles. We want to isotope S_1 further to a sphere disjoint from S_0; as just explained this will conclude the argument.

Let C be one of the circles of intersection, chosen in such a way that it bounds a 2-disc D_1 in S_1 not containing any other circles of intersection. In S_0, the circle C bounds two 2-discs D_0 and D_0'. One of the 2-spheres $D_0 \cup D_1$ and $D_0' \cup D_1$, say the former, bounds a 3-ball, as can be seen by considering the situation in the universal cover of $S^1 \times S^2$. This allows us to isotope S_1 across this 3-ball in order to remove the circle C of intersection. In the process, all circles of intersection contained in D_0 will be removed as well. See Figure 3 for a schematic picture. Iterating this procedure, we separate S_0 and S_1. □

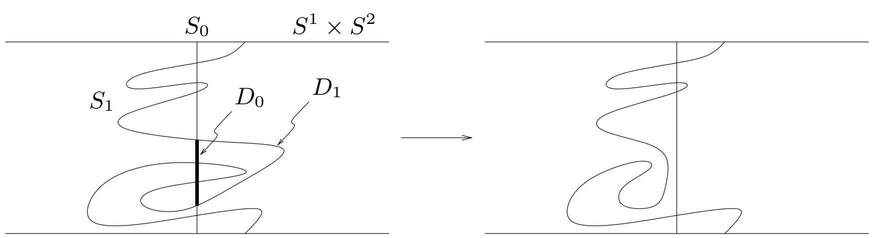

FIGURE 3. Removing intersections between S_0 and S_1.

5. FIXING A LEGENDRIAN CIRCLE

Now consider the oriented Legendrian circle $K_0 \subset (S^1 \times S^2, \xi_{\mathrm{st}})$ representing what we shall call the *positive* generator of $H_1(S^1 \times S^2)$, as shown in the front projection picture in Figure 4. (It is understood that $\mathbb{R}^3 \subset S^3$ be equipped with the standard contact structure $dz + x\,dy = 0$; Legendrian knots are illustrated in the front projection to the yz-plane.) This corresponds to the Legendrian circle on the left-hand side in Figure 1. In particular, that figure shows that the Legendrian knot $r_{\mathrm{c}}(K_0)$ is Legendrian isotopic to the positive stabilisation S_+K_0 of K_0.

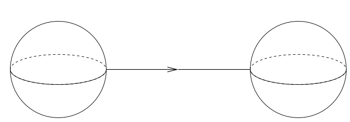

FIGURE 4. The Legendrian circle K_0.

Lemma 5. *For some $k \in \mathbb{Z}$, the contactomorphism $r_{\mathrm{c}}^k \circ f$ is contact isotopic to a contactomorphism fixing K_0.*

Proof. The image $f(K_0)$ will be some Legendrian knot representing the positive generator of $H_1(S^1 \times S^2)$ and, since f fixes S_0, going exactly once over the 1-handle. With the help of 'move 6' from [23], or what is also called the light bulb trick (cf. [12]), one can unknot $f(K_0)$ via a Legendrian isotopy (which extends to a contact isotopy by [18, Thm. 2.6.2]). An example is shown in Figure 5, where the final result of the isotopy is actually K_0. In general, the result will be some (multiple) stabilisation of K_0.

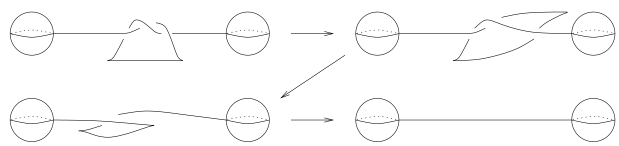

FIGURE 5. The light bulb trick used for unknotting.

Here is a more 'algorithmic' description of this unknotting procedure. First of all, by [23, Thm. 2.2] we may assume that, after a Legendrian isotopy, $f(K_0)$ is in standard form, i.e. its front projection is contained entirely between the two attaching balls for the 1-handle. Given a knotted piece of string with loose ends, one can clearly unknot it by contracting the string from one of its ends. If we imagine the attaching balls as the ends of such a piece of string, this contraction can be regarded as a motion of the right-hand ball, say. We thus remove all crossings in the front projection, while preserving the cusps; we need Legendrian Reidemeister moves of the second kind to slide all the cusps adjacent to the right-hand attaching ball over or under another strand in order to remove the crossing with that strand. With the light bulb trick this translates into a Legendrian isotopy with the attaching balls fixed. The final result of this Legendrian isotopy will be a Legendrian knot whose front projection has no crossings, but which now winds several times around the right-hand attaching ball in the yz-plane. One can bring the knot back into standard form (and still no crossings in the front projection) as follows: perform a move of type 6 to introduce a single kink in the front projection; then remove the kink with a Legendrian Reidemeister move of the first kind (see Figure 6); each such move reduces the (absolute) winding number of the front projection of $f(K_0)$ around the right-hand ball.

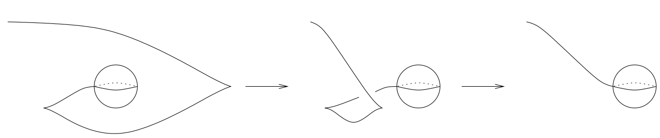

FIGURE 6. Reducing the winding number.

Positive and negative stabilisations can then be removed in pairs by a further application of the light bulb trick, as shown in Figure 7.

Thus, f is contact isotopic to a contactomorphism that maps K_0 to a stabilisation $S_{\pm}^n K_0$ for some $n \in \mathbb{N}_0$. Then $r_{\mathrm{c}}^{\mp n} \circ f$ is contact isotopic to a contactomorphism that fixes K_0. □

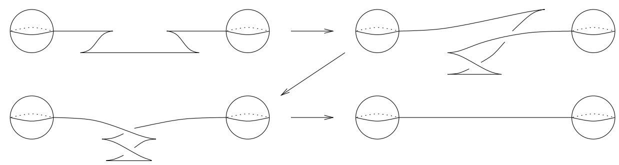

FIGURE 7. The light bulb trick used for removing stabilisations.

Remark. Even powers of r_{c} are isotopic to the identity, but not contact isotopic to the identity. This follows from the observation that the application of r_{c} to K_0 increases its rotation number by 1. Notice that ξ_{st} is trivial as a 2-plane bundle; a global non-vanishing section of ξ_{st} is given by

$$(x_2 - x_1)\partial_\theta + x_3\partial_{x_1} + x_3\partial_{x_2} - (x_1 + x_2)\partial_{x_3}.$$

So the rotation number is well defined for arbitrary Legendrian knots in the contact manifold $(S^1 \times S^2, \xi_{\mathrm{st}})$.

6. Fixing a neighbourhood of a Legendrian circle

We now want to show that after a further contact isotopy we may assume that $r_{\mathrm{c}}^k \circ f$ fixes a whole neighbourhood of K_0. We formulate this as a general statement.

Lemma 6. *Let K be a Legendrian knot in a contact 3-manifold (M, ξ), and let g be a contactomorphism of (M, ξ) that fixes K. Then g is contact isotopic to a contactomorphism that fixes a neighbourhood of K.*

Proof. By the tubular neighbourhood theorem for Legendrian knots [18, Cor. 2.5.9], we may identify a tubular neighbourhood $N(K)$ of K in (M, ξ) with a tubular neighbourhood of $S^1 \times \{0\}$ in $S^1 \times \mathbb{R}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2$ with contact structure $dz - y\,dx = 0$; the knot K is identified with $S^1 \times \{0\}$.

The dilatation

$$\delta_t(x, y, z) := (x, ty, tz)$$

is a contactomorphism of $S^1 \times \mathbb{R}^2$ for each $t \in \mathbb{R}^+$. This allows us to assume that the contactomorphic image of $N(K)$ in $S^1 \times \mathbb{R}^2$ has been chosen so large that when we restrict the contactomorphism induced by g to $S^1 \times D^2$, its image will stay inside this image of $N(K)$ (and that the same holds for the 1-parameter family of contact embeddings considered below). In fact, the argument in the proof of [7, Prop. 3.1] can be used to show that we may identify $N(K)$ contactomorphically with all of $S^1 \times \mathbb{R}^2$. Although this is not essential, we shall assume it for ease of notation. Thus, we think of (the restriction of) g as a contact embedding

$$g\colon\ S^1 \times D^2 \longrightarrow S^1 \times \mathbb{R}^2.$$

It will suffice to show that g is contact isotopic to the inclusion; the lemma then follows from the isotopy extension theorem for contact isotopies, cf. [18, Remark 2.6.8].

We now mimic the proof of the contact disc theorem [18, Thm. 2.6.7]. Write g in the form

$$(x, y, z) \longmapsto \big(X(x, y, z), Y(x, y, z), Z(x, y, z)\big).$$

The condition for this to be a *contact* embedding is

$$dZ - Y\,dX = \lambda(dz - y\,dx)$$

with some smooth function $\lambda\colon S^1 \times D^2 \to \mathbb{R}^+$. This can be rewritten as the following system of differential equations:

$$\begin{cases} \dfrac{\partial Z}{\partial x} - Y\dfrac{\partial X}{\partial x} & = & -\lambda y, \\ \dfrac{\partial Z}{\partial y} - Y\dfrac{\partial X}{\partial y} & = & 0, \\ \dfrac{\partial Z}{\partial z} - Y\dfrac{\partial X}{\partial z} & = & \lambda. \end{cases}$$

The assumption that g fixes K translates into

$$X(x,0,0) = x, \quad Y(x,0,0) = 0, \quad Z(x,0,0) = 0.$$

Now, for $t \in (0,1]$, consider the contact embedding

$$\delta_t^{-1} \circ g \circ \delta_t(x,y,z) = \bigl(X(x,ty,tz), \frac{1}{t}Y(x,ty,tz), \frac{1}{t}Z(x,ty,tz)\bigr).$$

For $t \to 0$, this converges to the map

$$g_0(x,y,z) := \bigl(x, y\cdot\frac{\partial Y}{\partial y}(x,0,0) + z\cdot\frac{\partial Y}{\partial z}(x,0,0), y\cdot\frac{\partial Z}{\partial y}(x,0,0) + z\cdot\frac{\partial Z}{\partial z}(x,0,0)\bigr).$$

From the above system of differential equations we deduce

$$\frac{\partial Z}{\partial y}(x,0,0) = 0, \quad \frac{\partial Z}{\partial z}(x,0,0) = \lambda(x,0,0) =: \lambda_0(x).$$

The first differential equation in the above system gives

$$\frac{\partial^2 Z}{\partial z\partial x} - \frac{\partial Y}{\partial z}\frac{\partial X}{\partial x} - Y\frac{\partial^2 X}{\partial z\partial x} = -\frac{\partial\lambda}{\partial z}\cdot y.$$

When we evaluate this at $(x,0,0)$, we find with the previous equations:

$$\lambda_0'(x) = \frac{\partial\lambda}{\partial x}(x,0,0) = \frac{\partial^2 Z}{\partial x\partial z}(x,0,0) = \frac{\partial Y}{\partial z}(x,0,0).$$

Finally, the first differential equation also yields

$$\frac{\partial^2 Z}{\partial y\partial x} - \frac{\partial Y}{\partial y}\frac{\partial X}{\partial x} - Y\frac{\partial^2 X}{\partial y\partial x} = -\frac{\partial\lambda}{\partial y}\cdot y - \lambda.$$

Since $\dfrac{\partial Z}{\partial y}(x,0,0) = 0$, we also have $\dfrac{\partial^2 Z}{\partial x\partial y}(x,0,0) = 0$. Thus, evaluating the foregoing equation at $(x,0,0)$ gives

$$\frac{\partial Y}{\partial y}(x,0,0) = \lambda_0(x).$$

In conclusion, we see that the map g_0 takes the form

$$g_0(x,y,z) = (x,\, y\cdot\lambda_0(x) + z\cdot\lambda_0'(x),\, z\cdot\lambda_0(x)).$$

It is easy to check that any map of this form (with $\lambda_0\colon S^1 \to \mathbb{R}^+$) is a contact embedding of $S^1 \times D^2$ into $S^1 \times \mathbb{R}^2$.

Our initial embedding g is thus seen to be contact isotopic to g_0, and the convex linear interpolation between λ_0 and the constant function 1 defines a contact isotopy between g_0 and the inclusion map. This finishes the proof of the lemma. $\square$

Remark. As explained in [18, Example 2.5.11], a universal model for the tubular neighbourhood of a Legendrian submanifold L in a higher-dimensional contact manifold is provided by a neighbourhood of the zero section $L \subset T^*L \subset \mathbb{R} \times T^*L$ in the 1-jet bundle of L with its canonical contact structure $dz - \lambda_{\mathrm{can}} = 0$, where λ_{can} is the canonical 1-form on T^*L, written in local coordinates $\mathbf{q}$ on L and dual coordinates $\mathbf{p}$ on the fibres of T^*L as $\lambda_{\mathrm{can}} = \mathbf{p}\, d\mathbf{q}$. The above proof carries over, *mutatis mutandis*, to show that the tubular neighbourhood of a Legendrian submanifold is unique not only up to contactomorphism, but up to contact isotopy.

Here is an alternative proof of Lemma 6. Admittedly, the methods used in it amount to cracking nuts with a sledgehammer, but they may be of some independent interest. We define a new contactomorphism h of (M, ξ) as follows, cf. [13, Remark 4.1]. Choose a standard tubular neighbourhood $N(K)$ of K, where the contact structure is given by $\cos\theta\, dx - \sin\theta\, dy = 0$ under the identification of $N(K)$ with $S^1 \times D^2$ (and K with $S^1 \times \{0\}$). Observe that $N(K)$ may be regarded as the space of (cooriented) contact elements of D^2.

Set $h = g$ on the closure of $M \setminus N(K)$. On a smaller tubular neighbourhood $N' \subset N(K)$, set $h = \mathrm{id}$. By the uniqueness up to contactomorphism of the non-rotative tight contact structure on $T^2 \times [0, 1]$ with two dividing curves on each boundary component, see [27], this h extends to a contactomorphism on all of (M, ξ).

Then $g^{-1} \circ h$ is a contactomorphism equal to the identity on $M \setminus N(K)$. So the restriction of $g^{-1} \circ h$ to $N(K)$ may be regarded as a contactomorphism, equal to the identity near the boundary, of the space of contact elements of D^2. According to a result of Giroux [21], the group of those contactomorphisms is connected. This gives a contact isotopy from $g^{-1} \circ h$ to the identity on M. It follows that g is contact isotopic to h, which has the desired properties.

Remark. Giroux's paper [21] has to be read with a certain amount of caution. Proposition 10 and the proofs of the main results (though not the results as such) are incorrect. The proofs can be fixed using the methods of [32].

7. Reduction to a space of contact elements

In a final step, we want to appeal once more to the result of Giroux [21] about contactomorphism groups of spaces of contact elements.

Lemma 7. *The complement of K_0 in $(S^1 \times S^2, \xi_{\mathrm{st}})$ is contactomorphic to the space of contact elements of $\mathbb{R}^2$.*

Proof. In [11] we described an explicit contactomorphism between the space of contact elements of $\mathbb{R}^2$ and the complement of a Legendrian unknot in S^3 with its standard contact structure (which we shall also write as ξ_{st}). That complement is seen to be contactomorphic to $(S^1 \times S^2 \setminus K_0, \xi_{\mathrm{st}})$ as follows.

An alternative surgery picture for $(S^1 \times S^2, \xi_{\mathrm{st}})$ is given by a single contact $(+1)$-surgery along a Legendrian unknot in (S^3, ξ_{st}). In this picture, K_0 becomes a Legendrian push-off of the surgery curve, see [12]. The cancellation lemma from [10], cf. [18, Prop. 6.4.5], says that contact (-1)-surgery along K_0 brings us back to (S^3, ξ_{st}). More specifically (as the proof of the cancellation lemma shows), K_0 may be regarded as the belt sphere of the surgery along the Legendrian unknot in (S^3, ξ_{st}), and the complement of that belt sphere in the surgered manifold is indeed contactomorphic to the complement of the surgery curve in the initial manifold. $\square$

In the preceding section we had found an integer k such that (after a contact isotopy) $r_{\mathrm{c}}^k \circ f$ fixes a neighbourhood of K_0. So we may interpret this map as a contactomorphism of the space of contact elements of D^2, equal to the identity near the boundary. By Giroux [21], this contactomorphism is contact isotopic (rel boundary) to the identity.

Thus, in total, our initial diffeomorphism f of $S^1 \times S^2$ (acting trivially on homology) has been shown to be isotopic to either id or r, as was claimed in Proposition 2.

Remarks. (1) The result of Giroux about the contactomorphism group of the space of contact elements of D^2 uses Cerf's theorem $\pi_0(\mathrm{Diff}^+(S^3)) = 0$ in its proof. So the described methods cannot, as yet, be used to give a contact geometric proof of Cerf's theorem. However, there is in fact a contact geometric proof of the slightly weaker form of Cerf's theorem, saying that every diffeomorphism of S^3 extends to a diffeomorphism of the 4-ball — a theorem popularly known as $\Gamma_4 = 0$. That proof is due to Eliashberg [14]; for an exposition see [18].

(2) Observe that our argument has shown the following: any contactomorphism of $(S^1 \times S^2, \xi_{\mathrm{st}})$ acting trivially on homology is contact isotopic to a uniquely determined integer power of r_{c}; any contactomorphism that is topologically isotopic to the identity is contact isotopic to an even power of r_{c}.

8. On the topology of the space of contact structures

Gonzalo and the second author have shown in [19] that there are essential loops in the space of contact structures on torus bundles over the circle. The main ingredient in that proof was the classification of contact structures on the 3-torus. Bourgeois [2] reproved their result with the help of contact homology and used that technique to detect higher non-trivial homotopy groups of the space of contact structures on a number of higher-dimensional manifolds. Here we formulate such a statement for the fundamental group of the space of contact structures on $S^1 \times S^2$.

Write Ξ_0 for the component of the space of contact structures on $S^1 \times S^2$ containing ξ_{st}, and Cont_0 for the subgroup of $\mathrm{Diff}_0 := \mathrm{Diff}_0(S^1 \times S^2)$ consisting of contactomorphisms of ξ_{st}. By Gray stability, we have a surjection

$$\begin{array}{rccc} \sigma\colon & \mathrm{Diff}_0 & \longrightarrow & \Xi_0 \\ & \phi & \longmapsto & T\phi(\xi_{\mathrm{st}}) \end{array}$$

with $\sigma^{-1}(\xi_{\mathrm{st}}) = \mathrm{Cont}_0$. As shown in [19], this gives rise to a long exact sequence

$$\ldots \stackrel{\Delta}{\longrightarrow} \pi_i(\mathrm{Cont}_0) \stackrel{\iota_\#}{\longrightarrow} \pi_i(\mathrm{Diff}_0) \stackrel{\sigma_\#}{\longrightarrow} \pi_i(\Xi_0) \stackrel{\Delta}{\longrightarrow} \pi_{i-1}(\mathrm{Cont}_0) \stackrel{\iota_\#}{\longrightarrow} \ldots,$$

where we write ι for the inclusion $\mathrm{Cont}_0 \to \mathrm{Diff}_0$; this is essentially the homotopy long exact sequence of a Serre fibration.

By the second remark at the end of the preceding section, we have $\pi_0(\mathrm{Cont}_0) \cong \mathbb{Z}$, generated by the contact isotopy class of r_{c}^2. Since this lies in the kernel of $\iota_\#$, there must be a subgroup isomorphic to $\mathbb{Z}$ in $\pi_1(\Xi_0)$. If we permit ourselves to rely on some additional information about the homotopy type of Diff_0, we can actually show this to be the full fundamental group of Ξ_0.

Proposition 8. *The component Ξ_0 of the space of contact structures on $S^1 \times S^2$ containing ξ_{st} has fundamental group isomorphic to $\mathbb{Z}$.*

Proof. The homotopy type of the group of homeomorphisms of $S^1 \times S^2$ was determined, modulo the Smale conjecture, by César de Sá and Rourke [5]. Hatcher's

proof [25] of the Smale conjecture not only completes their work, it also implies — as shown by Cerf [3] — that the space of diffeomorphisms of any 3-manifold is homotopy equivalent to its space of homeomorphisms. Thus,

$$\mathrm{Diff}_0(S^1 \times S^2) \simeq \mathrm{SO}_2 \times \mathrm{SO}_3 \times \Omega_0\mathrm{SO}_3,$$

where $\Omega_0\mathrm{SO}_3$ stands for the component of the contractible loop in the loop space of SO_3.

Now, $\pi_1(\Omega_0\mathrm{SO}_3) \cong \pi_2(\mathrm{SO}_3) = 0$, and the generators of $\pi_1(\mathrm{SO}_2)$ and $\pi_1(\mathrm{SO}_3)$ in the above factorisation of Diff_0 can be realised as loops of contactomorphisms

$$(\theta, \mathbf{x}) \longmapsto (\theta + \varphi, \mathbf{x}), \quad \varphi \in [0, 2\pi],$$

and

$$(\theta, \mathbf{x}) \longmapsto (\theta, r_\varphi(\mathbf{x})), \quad \varphi \in [0, 2\pi],$$

respectively. Thus, the homotopy exact sequence becomes

$$\pi_1(\mathrm{Cont}_0) \twoheadrightarrow \pi_1(\mathrm{Diff}_0) \to \pi_1(\Xi_0) \to \mathbb{Z} \to 0.$$

The proposition follows. □

9. Legendrian knots not distinguished by classical invariants

In [16], Fraser described an infinite family of Legendrian knots in the contact manifold

$$(M_0, \xi_0) := (S^1 \times S^2 \# S^1 \times S^2, \xi_{\mathrm{st}} \# \xi_{\mathrm{st}}),$$

all of which have the same topological knot type and the same classical invariants `tb` and `rot`, but which are nonetheless pairwise not Legendrian isotopic. The idea for distinguishing these knots is to perform Legendrian surgery on them (or contact (-1)-surgery in the language of [10]), and then to observe that the contact structures on the surgered manifold (which happens to be the 3-torus T^3) are pairwise not isotopic. This argument, in our view, is incomplete because it hinges on the statement "Legendrian surgery on Legendrian isotopic knots produces isotopic contact structures on the surgered manifold" — which is meaningless, as we want to explain.

Suppose you have two Legendrian isotopic knots L_0, L_1 in a contact 3-manifold (M, ξ). The Legendrian isotopy extends to a contact isotopy ϕ_t, $t \in [0, 1]$, of (M, ξ) with $\phi_1(L_0) = L_1$. For each $t \in [0, 1]$, the contactomorphism ϕ_t of (M, ξ) induces a contactomorphism between the contact manifold M_{L_0} obtained by Legendrian surgery along L_0 and the contact manifold $M_{\phi_t(L_0)}$ obtained by Legendrian surgery along $\phi_t(L_0)$. But there is no way, in general, to identify M_{L_0} with $M_{\phi_t(L_0)}$ (even as mere differential manifolds) other than with the diffeomorphism induced by ϕ_t. So we obtain a parametric family of contact manifolds, all of which are contactomorphic, but not an isotopy of contact structures on a fixed differential manifold.

In fact, in situations where there is a canonical way of identifying the surgered manifolds, the statement in question is false, in general. This is illustrated by the following example from [33, Exercise 11.3.12 (c)], see Figure 8. Contact (-1)-surgery on the 'shark' in (S^3, ξ_{st}) with its mouth on the left or on the right corresponds topologically to a surgery on the unknot with surgery coefficient -3 relative to the surface framing. If we take the obvious topological identification of the shark with the unknot, this allows us to identify the surgered manifold in both cases with the lens space $L(3, 1)$. With respect to this identification, the

two resulting contact structures on the surgered manifold $L(3,1)$ can be distinguished via their induced spin^c structure, so they are not isotopic. (Under the identification in question, which gives the two sharks the same, say the counterclockwise orientation, the shark on the left has $\mathtt{rot} = +1$, the one on the right, $\mathtt{rot} = -1$. This implies that the corresponding spin^c structures have first Chern class $c_1 = \pm 1 \in H^2(L(3,1);\mathbb{Z}) = \mathbb{Z}_3$; see [23, Prop. 2.3].) However, there is a Legendrian isotopy from one shark to the other (reversing its orientation), and this induces a contactomorphism of the surgered contact manifolds — it is simply the contactomorphism induced by the contactomorphism $(x,y,z) \mapsto (-x,-y,z)$ relating the two contact surgery diagrams.

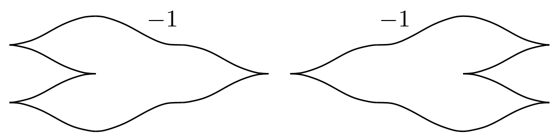

FIGURE 8. Contactomorphic, non-isotopic contact structures.

Thus, if one wants to show with the help of Legendrian surgery that two Legendrian knots L_0, L_1 cannot be Legendrian isotopic, one has to require, in general, that the surgered contact manifolds are not contactomorphic. In Fraser's set-up, unfortunately, the surgered manifolds happen to be contactomorphic by construction. Nonetheless, we now want to show that Fraser's idea can be made to work. In fact, the examples we are going to discuss presently are explicit realisations of the knots described only implicitly by Fraser.

Figure 9 shows a family L_k, $k \in \mathbb{Z}$, of Legendrian knots in (M_0, ξ_0); for $k < 0$, the zig-zags are to be interpreted as $|k|$ pairs of zig-zags in the opposite direction. Observe that $L_k = r_{\mathrm{c}}^k(L_0)$, where r_{c} is regarded as a contactomorphism acting only on the upper (in the picture) summand $S^1 \times S^2$ — there is a realisation of r_{c} that fixes a disc and hence is compatible with taking the connected sum. All these knots have the same topological knot type, as can be shown by applying the topological light bulb trick. Moreover, the well-known formulæ for computing the classical invariants — which take the same form for a Legendrian knot in 'standard form' in (M_0, ξ_0) as in (S^3, ξ_{st}), see [23] — give $\mathtt{tb}(L_k) = 1$ and $\mathtt{rot}(L_k) = 0$ for all $k \in \mathbb{Z}$ (for either orientation of those knots).

Theorem 9. *For $k \neq k'$, the knots L_k and $L_{k'}$ are not Legendrian isotopic.*

Proof. Arguing by contradiction, let us assume that there are two Legendrian isotopic knots L_{k_1} and L_{k_2}, with $k_1 \neq k_2$. Then $L_0 = r_{\mathrm{c}}^{-k_1}(L_{k_1})$ and $L_k = r_{\mathrm{c}}^{-k_1}(L_{k_2})$, where $k := k_2 - k_1$, will be Legendrian isotopic. Since $L_k = r_{\mathrm{c}}^k(L_0)$, this implies that r_{c}^k is contact isotopic to a contactomorphism of (M_0, ξ_0) that fixes L_0. By Lemma 6, r_{c}^k is then contact isotopic to a contactomorphism ϕ that fixes a neighbourhood $N(L_0)$ of L_0.

The Stein fillable and hence tight contact manifold obtained by contact (-1)-surgery on L_0 is T^3, see [24, Example 11.2.4], with its standard contact structure $\eta_1 := \ker(\sin\theta\, dx - \cos\theta\, dy)$, cf. [36]. Interpreted as a Kirby diagram, Figure 9 (with $k = 0$ and framing for the handle attachment equal to -1 relative to the contact framing) describes $T^2 \times D^2$. The D^2-fibre is represented by the cocore of

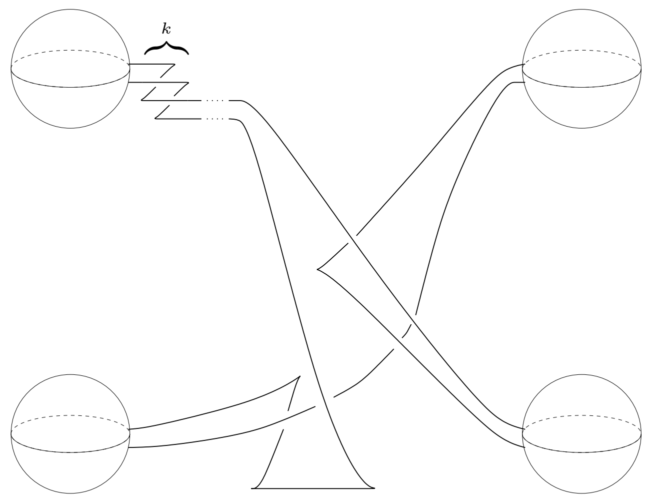

FIGURE 9. The Legendrian knots L_k.

the 2-handle, so the belt sphere of the surgery on L_0 is an S^1-fibre of T^3, see [24, Example 4.6.5].

This S^1-fibre corresponds (up to isotopy) to the θ-coordinate in the description of η_1, for the θ-circles are uniquely characterised by the fact that they become homotopically trivial in any Stein filling W of T^3, see the proof of [36, Lemma 4.3]. That fact rests on two observations. First of all, the homomorphism $H_1(T^3) \to H_1(W)$ induced by inclusion is surjective; this follows from the cell structure of Stein manifolds. Secondly, the θ-fibres must lie in the kernel of this homomorphism, otherwise one could pass to a cover and obtain a Stein filling of the contact structure $\eta_n := \ker(\sin(n\theta)\, dx - \cos(n\theta)\, dy)$ for some $n > 1$, which is impossible by a result of Eliashberg [15].

In the proof of the cancellation lemma given in [18, p. 323] it is shown explicitly that the belt sphere of the surgery is Legendrian isotopic, in the surgered manifold, to a Legendrian push-off of L_0. Alternatively, we can isotope it to a standard Legendrian meridian of L_0, as shown in [12, Prop. 2]. We now want to show that this standard Legendrian meridian is in fact *Legendrian* isotopic to the θ-fibre S^1_θ in (T^3, η_1). For this we appeal to the classification of linear Legendrian curves in (T^3, η_1) by Ghiggini [20]. The Thurston–Bennequin invariant $\mathtt{tb}(L)$ can be defined for such linear Legendrian curves L as the twisting of the contact structure relative to an incompressible torus containing L. This means that $\mathtt{tb}(S^1_\theta) = -1$, since the contact structure η_1 makes one negative twist along a θ-fibre relative to the framing given by the product structure $T^3 = S^1_\theta \times T^2_{x,y} = (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})^2$ (and the orientation $dy \wedge dx \wedge d\theta$ induced by η_1). This is the maximal $\mathtt{tb}$ in the topological knot type of S^1_θ; see [20, Thm. 5.4]. The definition of the rotation number $\mathtt{rot}$ for linear Legendrian curves in (T^3, η_1) depends on the choice of trivialisation of η_1, but it can be normalised so that $\mathtt{rot} = 0$ for curves realising the maximal $\mathtt{tb}$. According to [20, Thm. 2.5], the classical invariants suffice to classify Legendrian realisations of the topological knot type of S^1_θ. So all we have to show is that the standard Legendrian meridian μ_0 of L_0 has $\mathtt{tb}(\mu_0) = -1$ in the surgered manifold, i.e. in (T^3, η_1).

Now, μ_0 bounds a disc in S^3. With respect to the framing given by that disc in S^3, the contact structure makes one negative twist along μ_0 (since μ_0 is a standard Legendrian unknot in S^3 with $\mathtt{tb}(\mu_0) = -1$). A close inspection of [24, Example 4.6.5] shows that the framing which μ_0 inherits from the meridional disc, now regarded as a framing of μ_0 in T^3, is the same it inherits from an incompressible torus, whence it follows that $\mathtt{tb}(\mu_0) = -1$ also in the surgered manifold (T^3, η_1). (For that last statement about framings, imagine S^3 being cut in a plane passing through the attaching balls of one of the 1-handles, and with the attaching balls for the second 1-handle symmetric to this plane. Then, in the 2-sphere cut out by this plane, a circle around one of the attaching balls — that ball being seen as a disc in this 2-sphere — defines μ_0 up to isotopy. The 2-sphere with two discs removed, together with a cylinder contained in the boundary of the 1-handle, is an incompressible torus in the surgered manifold, containing μ_0.)

As a check for consistency, we observe that because of $\mathtt{tb}(S^1_\theta) = -1$ in (T^3, η_1), contact $(+1)$-surgery along such a fibre, which brings us back to (M_0, ξ_0) by the cancellation lemma, is topologically a surgery with framing given by the product structure of T^3, and that does indeed produce M_0.

If we perform the surgery along L_0 inside the neighbourhood $N(L_0)$, the fact that the belt sphere of the surgery is S^1_θ (up to Legendrian isotopy) implies that we have a contactomorphism between $(M_0 \setminus N(L_0), \xi_0)$ and $(T^3 \setminus N(S^1_\theta), \eta_1)$ for some neighbourhood $N(S^1_\theta)$ of S^1_θ. It follows that the contactomorphism ϕ of (M_0, ξ_0), which fixes $N(L_0)$, induces a contactomorphism of (T^3, η_1) that fixes $N(S^1_\theta)$. This may be interpreted as a contactomorphism of the space of contact elements of T^2 with a disc D^2 removed, equal to the identity near the boundary. By Giroux's theorem [21], this contactomorphism is contact isotopic (rel boundary) to one that is lifted from a diffeomorphism of the base $T^2 \setminus D^2$. (Recall that the differential of a diffeomorphism of any given manifold induces a contactomorphism of the space of contact elements of that manifold.) We continue to write ϕ for this contactomorphism and its extension to (M_0, ξ_0).

Using the action of the diffeotopy group of $T^2 \setminus D^2$ by contactomorphisms on $(T^3 \setminus N(S^1_\theta), \eta_1)$, we may assume that the identification of $(T^3 \setminus N(S^1_\theta), \eta_1)$ with $(M_0 \setminus N(L_0), \xi_0)$ has been chosen in such a way that one of the standard generators of $H_1(T^2 \setminus D^2)$ corresponds to a loop in $M_0 \setminus N(L_0)$ going once (homologically, or geometrically counted with sign) over the upper 1-handle in Figure 9.

A concrete Legendrian realisation K_1 of such a loop is shown in Figure 10, where S^1_θ is taken to be the fibre over $(x, y) = (1/2, 1/2)$. We take $\overline{K}_1 = \{y = y_0,\, \theta = 0\}$ (oriented by ∂_x, cooriented by $-\partial_y$); its Legendrian lift K_1 coincides with $\overline{K}_1$. Transverse to K_1 we see an annulus $\{x = 1/2\}$ in $T^3 \setminus N(S^1_\theta)$. Each of the two boundary components of that annulus bounds a disc in M_0, so there we have a 2-sphere transverse to K_1, corresponding to the S^2-factor in the upper summand $S^1 \times S^2$. We also write K_1 for the corresponding Legendrian loop in $(M_0 \setminus N(L_0), \xi_0)$.

The contactomorphism ϕ of (M_0, ξ_0), being isotopic to r_{c}^k, sends K_1 to a Legendrian knot $\phi(K_1) \subset M_0 \setminus N(L_0)$ that is smoothly homotopic in M_0 to K_1. This translates into a homotopy in $M_0 \setminus N(L_0)$ at the price of adding a meridional loop every time the original homotopy crosses L_0. In $T^3 \setminus N(S^1_\theta)$ this becomes a homotopy between K_1 and $\phi(K_1)$, modulo adding a θ-fibre for each of the meridional crossings. When projected to $T^2 \setminus D^2$, this defines a homotopy between $\overline{K}_1$ and the projection $\overline{K}_1'$ of $\phi(K_1)$.

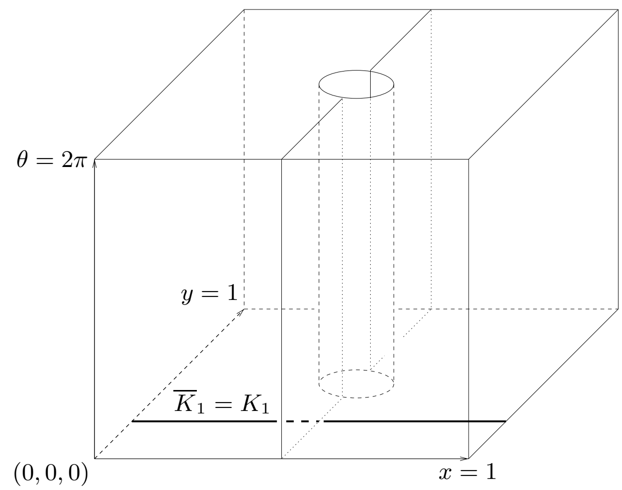

FIGURE 10. $T^3 \setminus N(S^1_\theta)$.

If we write $\overline{\phi}$ for the diffeomorphism of $T^2 \setminus D^2$ whose lift is ϕ, then $\overline{K}_1' = \overline{\phi}(\overline{K}_1)$, so $\overline{K}_1'$ is a simple closed curve in $T^2 \setminus D^2$ homotopic to $\overline{K}_1$. By Baer's theorem, see [35, 6.2.5], there is an isotopy of $T^2 \setminus D^2$, identical near the boundary, that moves $\overline{K}_1'$ back to $\overline{K}_1$ (with the original orientation, for homological reasons). The lift of this isotopy is a contact isotopy of $(T^3 \setminus N(S^1_\theta), \eta_1)$, fixed near the boundary, that moves $\phi(K_1)$ back to K_1.

Thus, r_{c}^k is contact isotopic to a contactomorphism of (M_0, ξ_0) that fixes K_1, which means that r_{c}^k sends K_1 to a Legendrian isotopic copy of K_1, contradicting the fact that r_{c}^k changes the rotation number of any oriented Legendrian circle that passes once (in positive direction, passings counted with sign) over the upper 1-handle by k. $\square$

Remark. Prior to Fraser's work, no examples were known of Legendrian knots that could not be distinguished by the classical invariants. The first examples of this type in $(\mathbb{R}^3, \xi_{\mathrm{st}})$ were found by Chekanov [6], who used Legendrian contact homology to distinguish the knots. Various other non-classical invariants have been developed in the meantime, such as normal rulings [8, 17] or knot Floer homology invariants [31, 34].

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